Original system
$A\vec{x}=\vec{b}$

$$\begin{bmatrix} 1 & 2 & 1 \\ 4 & 5 & 10 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 4 \\ 10 \end{bmatrix}$$

Solutions are:

$$\begin{array}{cccc} \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, & \begin{bmatrix} 5 \\ 0 \\ -1 \end{bmatrix}, & \begin{bmatrix} -5 \\ 4 \\ 1 \end{bmatrix}, & \dots \\ \text{BFS} & \text{Basic non-feasible solution} & \text{non-basic non-feasible solution} & \end{array}$$

Consider

$$\left[\begin{array}{ccc|cc} 1 & 2 & 1 & 1 & 0 \\ 4 & 5 & 10 & 0 & 1 \end{array}\right]\left[\begin{array}{c} x \\ y \\ z \\ \hdashline a \\ b \end{array}\right]=\begin{bmatrix} 4 \\ 10 \end{bmatrix}$$

$\left.\begin{array}{c} a \\ b \end{array}\right\}$ artificial variables

augmented system
$[A \mid I]\begin{bmatrix} \vec{x} \\ \vec{x}_a \end{bmatrix}=\vec{b}$

$\Leftrightarrow A\vec{x}+I\vec{x}_a=\vec{b}$

Solutions are

$$\begin{array}{cccc} \left[\begin{array}{c} 2 \\ 1 \\ 0 \\ \hdashline 0 \\ 0 \end{array}\right], & \left[\begin{array}{c} 5 \\ 0 \\ -1 \\ \hdashline 0 \\ 0 \end{array}\right], & \left[\begin{array}{c} -5 \\ 4 \\ 1 \\ \hdashline 0 \\ 0 \end{array}\right], & \\ \text{BFS} & \text{BnFS} & \text{nBnFS} & \end{array}$$

of original problem ($A\vec{x}=\vec{b}$).

and

$$\begin{array}{ccc} \left[\begin{array}{c} 0 \\ 0 \\ 0 \\ \hdashline 4 \\ 10 \end{array}\right] & \left[\begin{array}{c} 0 \\ 0 \\ 1 \\ \hdashline 3 \\ 0 \end{array}\right] & \left[\begin{array}{c} 1 \\ 0 \\ 0 \\ \hdashline 3 \\ 6 \end{array}\right] \\ \text{BFS} & \text{BFS} & \text{nBFS} \end{array}$$

of augmented problem $[A|I]\begin{bmatrix} \vec{x} \\ \vec{x}_b \end{bmatrix}=\vec{b}$

but <u>NOT</u> original problem $A\vec{x}=\vec{b}$

"How to do this"
??

"$[A|I]\begin{bmatrix} \vec{x} \\ \vec{x}_a \end{bmatrix}=\vec{b} \Rightarrow A\vec{x}=\vec{b}$" if $\vec{x}_a=\vec{0}$

Augmented solution is solution to original problem if all artificial variables $\equiv 0$

Eg 3.4

$$x_0 = x_1 + x_2$$

$$\begin{cases} 2x_1 + x_2 \geq 4 \\ x_1 + 2x_2 = 6 \\ x_1, x_2 \geq 0 \end{cases}$$

Not in FCF

$\Rightarrow$ Standard form

$$x_0 = x_1 + x_2 - 0x_3$$

$$\begin{cases} 2x_1 + x_2 - x_3 = 4 \\ x_1 + 2x_2 = 6 \\ x_1, x_2, x_3 \geq 0 \end{cases} \qquad A = \begin{pmatrix} 2 & 1 & -3 \\ 1 & 2 & 0 \end{pmatrix}$$

Try $B = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 4 \\ 6 \end{pmatrix} \Rightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2/3 \\ 8/3 \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2/3 \\ 8/3 \\ 0 \end{pmatrix} \text{ a BFS}$$

Try $B = \begin{pmatrix} 1 & -3 \\ 2 & 0 \end{pmatrix}$

$$\begin{pmatrix} 1 & -3 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 6 \end{pmatrix} \Rightarrow \begin{pmatrix} x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3 \\ -1/3 \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ -1/3 \end{pmatrix}$$ Basic solution but not feasible

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M-method: artificially create an identity matrix

$$x_0 = x_1 + x_2 - 0x_3 - Mx_4 - Mx_5$$

$$\begin{cases} 2x_1 + x_2 - x_3 + x_4 = 4 \\ x_1 + 2x_2 + x_5 = 6 \\ x_1 \quad x_2 \quad x_3 \quad x_4 \quad x_5 \geq 0 \end{cases} \qquad \text{new } A = \left(\begin{array}{ccc|cc} 2 & 1 & -3 & 1 & 0 \\ 1 & 2 & 0 & 0 & 1 \end{array}\right)$$

(i) obviously $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 4 \\ 6 \end{pmatrix}$ is a starting BFS for the augmented artificial problem, but not of the original problem

(ii) the cost $-M$ will drive x_4 & x_5 to zero so that at optimum

$$\underbrace{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} * \geq 0 \\ * \geq 0 \\ * \geq 0 \\ 0 \\ 0 \end{pmatrix}}_{\text{BFS}} \Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} * \geq 0 \\ * \geq 0 \\ * \geq 0 \end{pmatrix}$$ is the optimal solution of the original problem